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Finite Extremal Characterization of Strong Uniqueness in Normed Spaces

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A strongly unique best approximation m in a finite-dimensional subspace M of a real normed linear space X to an element $x \in X \setminus M$ is characterized by means of a finite number of extremal points of the closed unit ball in the dual space X^*. This result is applied to weak Chebyshev subspaces in $C(T)$. © 1990 Academic Press, Inc.

1. INTRODUCTION

Let $M \neq \{0\}$ be a finite-dimensional linear subspace of a real normed linear space X, and let x be an element of $X \setminus M$. We recall that an element $m \in M$ is called a *strongly unique best approximation* in M to x if there exists a constant $c > 0$ such that

$$\|x - m\| \leqslant \|x - y\| - c\,\|m - y\| \tag{1.1}$$

for all y in M. A first dual characterization of strongly unique best approximations is due to Wulbert [10, 11]. More precisely, if F is the duality mapping [2, Section 1.2.4] of $X \setminus \{0\}$ into the family of all non-empty w^*-compact convex subsets of the dual space X^* of X defined by

$$F(z) = \{f \in X^* : \|f\| = 1 \text{ and } f(z) = \|z\|\}, \qquad z \in X \setminus \{0\}, \tag{1.2}$$

then we have

WULBERT'S THEOREM. *An element $m \in M$ is a strongly unique best approximation in M to $x \in X \setminus M$ if and only if*

$$\sup_{f \in F(x-m)} f(y) > 0$$

for all $y \neq 0$ in M.

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In view of the Krein–Milman Theorem, one can easily deduce that the Wulbert Theorem remains true if we replace $F(x-m)$ by the set

$$\mathrm{Ext}[F(x-m)] = F(x-m) \cap \mathrm{Ext}[B(X^*)], \tag{1.3}$$

where $\mathrm{Ext}[A]$ is the set of all extremal points of a set A and $B(X^*)$ is the closed unit ball in X^*. The sets $F(x-m)$ and $\mathrm{Ext}[F(x-m)]$ can be uncountable, which is very unfavourable in applications of strong uniqueness. Therefore, in this paper we characterize a strongly unique best approximation m in M by means of a finite number of functionals from the set $\mathrm{Ext}[F(x-m)]$. This characterization is both a counterpart of the finite dual characterization of best approximations and a refinement of the characterization of strongly unique best approximations by elements of finite dimensional subspaces M in the space $X=C(T)$ due to Singer [9, Theorem 1.11] and Bartelt and McLaughlin [3, Theorem 6], respectively.

2. Main Results

Let us suppose additionally that the dimension of subspace M is equal to $n \geqslant 1$. A sequence of functionals $(f_i)_0^n$ in X^* is said to be *linearly sgn-dependent on M* if

$$\left(\sum_{i=0}^{n} \alpha_i f_i\right)(M) = \{0\} \Rightarrow \operatorname{sgn} \alpha_0 = \cdots = \operatorname{sgn} \alpha_n, \tag{2.1}$$

where $\operatorname{sgn} \alpha = 0$ if $\alpha = 0$ and $\operatorname{sgn} \alpha = \alpha/|\alpha|$ if $\alpha \neq 0$. In the following, we shall use the symbols δ_{ij}, $\operatorname{span}(A)$ and $\operatorname{co}(A)$ to denote the Kronecker delta, the linear space spanned by a subset A of X and the convex hull of A, respectively.

Lemma 2.1. *A sequence of functionals $(f_i)_0^n$ in X^* is linearly sgn-dependent on an n-dimensional linear subspace M of X if and only if the sequence $(f_i)_1^n$ is linearly independent on M and*

$$f_0(m_i) < 0; \qquad i = 1, ..., n, \tag{2.2}$$

where $(m_i)_1^n$ is the basis in M such that

$$f_i(m_j) = \delta_{ij}; \qquad i, j = 1, ..., n. \tag{2.3}$$

Proof. If a sequence $(f_i)_0^n$ in X^* is linearly sgn-dependent on M, then functionals $f_1, ..., f_n$ are linearly independent on M. Indeed, if we have

$$\left(\sum_{i=0}^{n} \alpha_i f_i\right)(M) = \{0\} \tag{2.4}$$

with $\alpha_0 = 0$, then (2.1) implies that each α_i is equal to zero. Hence there exists a basis $(m_i)_1^n$ in M satisfying conditions (2.3). Since $\dim M^* = \dim M$, it follows that functionals $f_0, ..., f_n$ are linearly dependent on M. This means that identity (2.4) holds for some α_i, not all equal to zero. By (2.1) we have

$$\operatorname{sgn} \alpha_0 = \cdots = \operatorname{sgn} \alpha_n \neq 0. \tag{2.5}$$

Moreover, inserting $m_j \in M$ into (2.4) and using (2.3), we get

$$\alpha_0 f_0(m_j) + \alpha_j = 0; \qquad j = 1, ..., n. \tag{2.6}$$

This in conjunction with (2.5) gives (2.2), which completes the proof of necessity. Conversely, suppose that conditions (2.2)–(2.4) are satisfied. Then it follows from (2.6) that implication (2.1) is true, which completes the proof. ∎

Now one can proceed to establish a finite extremal characterization of strongly unique best approximations, which is the main result of this paper. This characterization uses the notion of the *algebraic interior of* $\operatorname{co}\{g_1, ..., g_k\}$ ($g_i \in X^*$, $k \geq 1$) which consists of all functionals f of the form

$$f = \sum_{i=1}^{k} \lambda_i g_i,$$

where $\lambda_i > 0$ for all i and $\lambda_1 + \cdots + \lambda_k = 1$. Note that the algebraic interior of the set $\operatorname{co}\{g_1\}$ is equal to $\{g_1\}$.

THEOREM 2.1. *An element $m \in M$ is a strongly unique best approximation in an n-dimensional linear subspace M of a real normed linear space X to an element $x \in X \backslash M$ if and only if there exist functionals $f_1, ..., f_n, g_1, ..., g_k \in$ $\operatorname{Ext}[F(x-m)]$ $(1 \leq k \leq n)$ such that $(f_i)_0^n$ is linearly sgn-dependent on M for some functional f_0 in the algebraic interior of* $\operatorname{co}\{g_1, ..., g_k\}$.

Proof. If there exists a sequence $(f_i)_0^n$ of linearly sgn-dependent on M functionals in the convex hull $\operatorname{co}(\operatorname{Ext}[F(x-m)])$, then by Lemma 2.1 we have

$$\max\{f_0(y), f_1(y), ..., f_n(y)\} = \max\left\{\sum_{i=1}^{n} \alpha_i f_0(m_i), \alpha_1, ..., \alpha_n\right\} > 0$$

for all $y = \sum_{i=1}^{n} \alpha_i m_i \neq 0$ in M. This in conjunction with the Wulbert Theorem implies that m is a strongly unique best approximation in M to x. Conversely, suppose that m is a strongly unique best approximation in

$M = \operatorname{span}\{x_1, ..., x_n\}$ to $x \in X \backslash M$. Since this is equivalent to the fact that 0 is a strongly unique best approximation in M to $z = x - m \neq 0$, we may assume without loss of generality that $m = 0$. Hence we conclude, as in the proof of Bartelt and McLauglin's theorem [3, Theorem 6], that $0 \in \operatorname{co}(\operatorname{Ext}[T(x)])$ but $0 \notin \operatorname{Ext}[T(x))]$, where the compact set $T(x) \subset \mathbb{R}^n$ is the image of the w^*-compact convex set $F(x) \subset X^*$ under the linear w^*-continuous mapping

$$U: X^* \ni f \to (f(x_1), ..., f(x_n)) \in \mathbb{R}^n.$$

Therefore, it follows from the Caratheodory Theorem [4, p. 17] that $0 \in \mathbb{R}^n$ can be expressed in the form

$$0 = \sum_{i=0}^{p} \lambda_i \gamma_i; \qquad \gamma_i \in \operatorname{Ext}[T(x)], \tag{2.7}$$

where $\lambda_i > 0$ for all i, $\lambda_0 + \cdots + \lambda_p = 1$ and p $(1 \leqslant p \leqslant n)$ is the minimal integer for which identity (2.7) holds. By the proof of Caratheodory Theorem [4], the minimality of p implies that exactly p points, say $\gamma_1, ..., \gamma_p$, are linearly independent. Multiplying both sides of (2.7) by vector $\alpha = (\alpha_1, ..., \alpha_n)$, and next using the fact that $\gamma_i = (f_{1i}(x_1), ..., f_{1i}(x_n))$ for some functional f_{1i} in $\operatorname{Ext}[F(x)]$ (since $\operatorname{Ext}[T(x)] = \operatorname{Ext}[U(F(x))] \subset U(\operatorname{Ext}[F(x)])$, see [7, p. 401]), we obtain

$$\sum_{i=0}^{p} \lambda_i f_{1i}(y) = 0 \tag{2.8}$$

for all $y = \sum_{i=1}^{n} \alpha_i x_i \in M$. By minimality of p and linearity of U we conclude that functionals $f_{11}, ..., f_{1p}$ are linearly independent on M. Clearly, these functionals are also linearly independent on any p-dimensional linear subspace L_1 such that $M = L_1 \oplus M_1$ is a direct sum of L_1 and the $(n-p)$-dimensional subspace M_1 defined by

$$M_1 = \bigcap_{i=1}^{p} \{y \in M: f_{1i}(y) = 0\} = \bigcap_{i=0}^{p} \{y \in M: f_{1i}(y) = 0\}, \tag{2.9}$$

where the last equality follows directly from (2.8). Moreover, if $m_{11}, ..., m_{1p}$ is a basis of L_1 satisfying the conditions

$$f_{1i}(m_{1j}) = \delta_{ij}; \qquad i, j = 1, ..., p,$$

then in view of (2.8) we get $f_{10}(m_{1j}) = -\lambda_j/\lambda_0 < 0$ for all j. Hence by Lemma 2.1 functionals $f_{10}, f_{11}, ..., f_{1p}$ are linearly sgn-dependent on L_1, which completes the proof in the case when $p = n$, i.e., when $L_1 = M$ and $M_1 = \{0\}$. In particular, it follows that the necessity part of our theorem is true in the case when $n=1$. Now one can complete the proof by inducing on $n=\dim M$. Indeed, suppose that the necessity is true for any subspace of dimension less than n. Since $m=0$ is a strongly unique best approximation in M to $x \in X \backslash M$, it follows from (1.1) that $m=0$ is also a strongly unique best approximation in the q-dimensional subspace M_1 defined by (2.9) to the element x, where $q=n-p<n$. Clearly, we can assume that $q \geqslant 1$ and apply the induction hypothesis to M_1 in order to get functionals $f_{21}, \ldots, f_{2q}, g_1, \ldots, g_k \in \operatorname{Ext}[F(x)]$ $(1 \leqslant k \leqslant q)$ such that $(f_{2i})_0^q$ is linearly sgn-dependent on M_1 for some functional f_{20} in the algebraic interior of $\operatorname{co}\{g_1, \ldots, g_k\}$. Now Lemma 2.1 implies that $(f_{2i})_1^q$ is linearly independent on M_1 and hence on M. Define the space L_1 of dimension p by

$$L_1=\bigcap_{i=1}^{q}\{y \in M: f_{2i}(y)=0\}. \tag{2.10}$$

Since $L_1 \cap M_1=\{0\}$, we have $M=L_1 \oplus M_1$, and so functionals $f_{10}, \ldots, f_{1p}$ are linearly sgn-dependent on L_1. Consequently one can choose bases $(m_{1j})_1^p$ and $(m_{2j})_1^q$ for L_1 and M_1, respectively, so that $f_{vi}(m_{vj})=\delta_{ij}$ $(v=1,2)$ for all i, j. By (2.9) (2.10) functionals $f_{11}, \ldots, f_{1p}, f_{21}, \ldots, f_{2q}$ are linearly independent on $M=L_1 \oplus M_1$, and

$$f_{vi}(m_{\mu j})=\delta_{v\mu}\, \delta_{ij}; \qquad v, \mu=1,2, \tag{2.11}$$

for all i, j. On the other hand, by (2.8)–(2.9) we have $f_{10}=0$ on M_1. Moreover, in view of Lemma 2.1, we get $f_{10}(m_{1s})<0$ and $f_{20}(m_{2r})<0$, whenever $1 \leqslant s \leqslant p$ and $1 \leqslant r \leqslant q$. Thus for each $\vartheta>0$ sufficiently small, the functional $f_0=(1-\vartheta) f_{10}+\vartheta f_{20}$ satisfies $f_0(m_{2r})=\vartheta f_{20}(m_{2r})<0$ and

$$f_0(m_{1s})=(1-\vartheta) f_{10}(m_{1s})+\vartheta f_{20}(m_{1s})<0.$$

This in conjunction with (2.11), Lemma 2.1, and the fact that f_{20} belongs to the algebraic interior of $\operatorname{co}\{g_1, \ldots, g_k\}$ implies that functionals $f_0, f_{11}, \ldots, f_{1p}, f_{21}, \ldots, f_{2q}$ are linearly sgn-dependent on M, where f_0 is a convex combination of $k+1 \leqslant n$ elements $f_{10}, g_1, \ldots, g_k \in \operatorname{Ext}[F(x)]$ with positive coefficients. ∎

Let us note that a corollary of Theorem 2.1 is the following "0 in the convex hull" characterization of strongly unique best approximations, which is a counterpart of the "0 in the convex hull" characterization of best approximations due to Singer [9, Theorem 1.11].

COROLLARY 2.1. *An element $m \in M$ is a strongly unique best approximation in an n-dimensional linear subspace M of a real normed linear space X to an element $x \in X \backslash M$ if and only if there exist functionals $f_1, \ldots, f_{n+k} \in$*

$\mathrm{Ext}[F(x-m)]$ $(1 \leqslant k \leqslant n)$ *and positive numbers* $\lambda_1, ..., \lambda_{n+k}$ *with* $\lambda_1 + \cdots + \lambda_{n+k} = 1$ *such that* $(f_i)_1^n$ *is linearly independent on* M *and*

$$\sum_{i=1}^{n+k} \lambda_i f_i(y) = 0 \tag{2.12}$$

for all y *in* M.

Proof. If m is a strongly unique best approximation in M to $x \in X \setminus M$ then it follows from the definition of sgn-dependence and Theorem 2.1 that (2.12) holds, whenever functionals $f_1, ..., f_n, f_{n+1} = g_1, ..., f_{n+k} = g_k$ are as in Theorem 2.1. Moreover, by Lemma 2.1 the sequence $(f_i)_1^n$ is linearly independent on M. On the other hand, we can set

$$f_0 = \sum_{i=n+1}^{n+k} \beta_i f_i$$

with $\beta_i = \lambda_i/(\lambda_{n+1} + \cdots + \lambda_{n+k})$. By (2.12) we have $f_0(m_j) < 0$ for $j = 1, ..., n$, where $(m_j)_1^n$ is the basis of M defined by conditions (2.3). Hence one can apply Lemma 2.1 and Theorem 2.1 in order to complete the proof. ▌

It is clear that this corollary provides much more precise characterization of strong uniqueness than the Wulbert Theorem and Theorem 6 of Bartelt and McLaughlin [3]. A usefulness of Theorem 2.1 and Corollary 2.1 depends on the structure of sets $\mathrm{Ext}[F(z)] = F(z) \cap \mathrm{Ext}[B(X^*)]$ with $z \neq 0$ in X. This structure is especially simple in the case when $X = C(T)$ is the Banach space of all continuous real-valued functions defined on a compact Hausdorff space T with the uniform norm. Indeed, by the well-known characterization [5, p. 441, Lemma 6] of $\mathrm{Ext}[B(C^*(T))]$, we have

$$\mathrm{Ext}[F(z)] = \{\mathrm{sgn}[z(t)]\, \delta_t : t \in \mathrm{ext}(z)\}$$

for all $z \neq 0$ in $C(T)$, where $\mathrm{ext}(z) = \{t \in T: |z(t)| = \|z\|\}$ and functionals δ_t on $C(T)$ are defined by $\delta_t y = y(t)$, $y \in C(T)$. Clearly, a sequence of functionals $\sigma_1 \delta_{t_1}, ..., \sigma_{n+k} \delta_{t_{n+k}}$ $(\sigma_i = \mathrm{sgn}[z(t_i)],\ t_i \in T)$ contains n functionals linearly independent on $M = \mathrm{span}\{x_1, ..., x_n\} \subset C(T)$ if and only if the rank of matrix $[x_j(t_i)] = [x_j(t_i)]_{j=1,\ i=1}^{n,\ n+k}$ is equal to n. Thus Corollary 2.1 yields

COROLLARY 2.2. *A function* $m \in M$ *is a strongly unique best approximation in an n-dimensional subspace* $M = \mathrm{span}\{x_1, ..., x_n\}$ *of* $C(T)$ *to a function* $x \in C(T) \setminus M$ *if and only if there exist points* $t_1, ..., t_{n+k} \in \mathrm{ext}(z)$ $(1 \leqslant k \leqslant n$ *and* $z = x - m)$ *such that* $\mathrm{rank}[x_j(t_i)] = n$ *and*

(A) *the system of linear equations*

$$\sum_{i=1}^{n+k} \lambda_i \operatorname{sgn}[z(t_i)]\, x_j(t_i) = 0; \qquad j = 1, ..., n,$$

has a positive solution $\lambda_1, ..., \lambda_{n+k}$.

Now suppose additionally that T is a compact subset of the real line. Then an n-dimensional subspace $M = \operatorname{span}\{x_1, ..., x_n\}$ of $C(T)$ is called *weak Chebyshev*, if there exists an integer $\sigma \in \{-1, 1\}$ such that $\sigma \det[x_j(t_i)] \geqslant 0$ for all points $t_1 < \cdots < t_n$ in T. For such subspaces M, the necessity part of Corollary 2.2 can be established in a more precise form.

COROLLARY 2.3. *A function $m \in M$ is a strongly unique best approximation in an n-dimensional weak Chebyshev subspace $M = \operatorname{span}\{x_1, ..., x_n\}$ of $C(T)$ to a function $x \in C(T)\backslash M$ if and only if there exist points $t_1 < \cdots < t_{n+k}$ in $\operatorname{ext}(z)$ $(1 \leqslant k \leqslant n$ and $z = x - m)$ for which $\operatorname{rank}[x_j(t_i)] = n$ and conditions* (A) *and*

(B) *there exist integers $1 \leqslant k_0 < \cdots < k_n \leqslant n + k$ such that*

$$z(t_{k_{i-1}})\, z(t_{k_i}) < 0; \qquad i = 1, ..., n,$$

are satisfied.

Proof. From Theorem 7 of Gantmacher and Krein [6, Section 5.2] it follows that any solution $\lambda_1 \operatorname{sgn}[z(t_1)], ..., \lambda_{n+k} \operatorname{sgn}[z(t_{n+k})]$ $(\lambda_i > 0)$ of the system of linear equations given in (A) has at least n sign changes. Hence one can apply Corollary 2.2 to finish the proof. ∎

COROLLARY 2.4. *A function $m \in M$ is a strongly unique best approximation in an n-dimensional weak Chebyshev subspace $M = \operatorname{span}\{x_1, ..., x_n\}$ of $C(T)$ to a function $x \in C(T)\backslash M$ if and only if there exist points $t_1 < \cdots < t_{n+k}$ in $\operatorname{ext}(z)$ $(1 \leqslant k \leqslant n$ and $z = x - m)$ such that conditions* (A) *and*

(C) *there exist integers $1 \leqslant k_1 < \cdots < k_n \leqslant n + k$ such that*

$$\det[x_j(t_{k_i})] \neq 0 \qquad \text{and} \qquad z(t_{k_{i-1}})\, z(t_{k_i}) < 0; \qquad i = 2, ..., n,$$

are satisfied.

Proof. In view of Corollary 2.3 it is sufficient to prove condition (C) under the assumption that m is a strongly unique best approximation in M to $x \in C(T)\backslash M$. For this purpose, we apply Corollary 2.3 to show that the vector $(m(t_1), ..., m(t_{n+k}))$ is a strongly unique best approximation in $M_{n,k}$ to $(x(t_1), ..., x(t_{n+k}))$, where $M_{n,k} \subset C(S)$ is the n-dimensional weak

Chebyshev subspace of all functions in M with domains restricted to the set $S=\{t_1, ..., t_{n+k}\}$ of points t_i defined as in Corollary 2.3. Now, if condition (C) is not satisfied then one can construct a vector $(y_1, ..., y_{n+k}) \in M_{n,k}\setminus\{0\}$ such that $[x(t_i)-m(t_i)]\, y_i \leqslant 0$ for all i, which leads to a contradiction with the Wulbert Theorem. We omit details of this construction, since it is already given in [8, pp. 27–30]; but note that the Nürnberger's construction can be considerably simplified, since condition card $(S)=n+k$ implies that all minima and maxima occurring in this construction are attained. ∎

In order to compare Corollaries 2.3–2.4 with a celebrated theorem due to Nürnberger [8, Theorem 1.4], we first recall that linearly ordered disjoint subsets $T_1 < \cdots < T_j$ of ext(z) are called *alternating extremal sets of* z, if $z(t_{i-1})\, z(t_i)<0$ $(i=2, ..., j)$ for all points $t_p \in T_p$ $(p=i-1, i)$. Next, we divide points $t_1, ..., t_{n+k} \in \text{ext}(z)$ occurring in Corollaries 2.3–2.4 into $n+p$ extremal sets T_i:

$$\{t_1, ..., t_{n+k}\} = T_1 \cup \cdots \cup T_{n+p}.$$

Clearly, by condition (B) we have $1 \leqslant p \leqslant k \leqslant n$. Now one can compare Corollaries 2.3–2.4 with [8, Theorem 1.4] and derive the following conclusions:

(a) The number of alternating extremal sets T_i is less than or equal to $2n$ in Corollaries 2.3–2.4, while it is only finite in Theorem 1.4.

(b) Each set T_i may consist of at most n elements and it may be infinite, respectively. Therefore, our corollaries give the first finite extremal characterization of strong uniqueness for weak Chebyshev subspaces.

(c) It is striking that Corollaries 2.3–2.4 enable us to verify the strong uniqueness by examining only sequences of $n+k$ $(1 \leqslant k \leqslant n)$ points from ext(z).

(d) A rather complicated determinant condition (2b) presented in Theorem 1.4 is replaced by condition (A) in Corollaries 2.3–2.4 which can be easily verified by using computers.

(e) The proof of Corollaries 2.3–2.4 is comparatively very simple and short.

Now, following Ault *et al.* [1] suppose that $M=\text{span}\{x_1, ..., x_n\}$ is an *interpolating subspace* of dimension n of a real normed linear space X. This assumption is equivalent [1, Theorem 2.1] to the fact that

$$D_0 := \det[f_i(x_j)] \neq 0$$

for each set of n linearly independent functionals $f_1, ..., f_n$ in Ext$[B(X^*)]$. Hence it follows that functionals $f_{10}, ..., f_{1p} \in F(x) \cap \text{Ext}[B(X^*)]$ $(1 \leqslant$

$p \leqslant n$) constructed in the proof of our Theorem 2.1 are linearly dependent on M if and only if $p=n$. Thus we have $p=n$ in (2.8) and $k=1$ in Theorem 2.1. Additionally, by the Cramer's rule the elements $(m_i)_1^n$ of M, defined by the interpolating conditions (2.3), are equal to

$$m_i = \sum_{j=1}^{n} (-1)^{i+j} (D_{ij}/D_0) x_j,$$

where D_{ij} is the minor of $f_0(x_j)$ in $D_i = \det[f_\nu(x_\mu)]_{\nu=0,\ \mu=1}^{n,\ \ n}$ with $\nu \neq i$. Therefore, by applying Lemma 2.1 and Theorem 2.1, we get

COROLLARY 2.5. *An element $m \in M$ is a strongly unique best approximation in an n-dimensional interpolating subspace $M = \operatorname{span}\{x_1, ..., x_n\}$ of a real normed linear space X to an element $x \in X \setminus M$ if and only if there exist functionals $f_0, ..., f_n$ in* $\operatorname{Ext}[F(x-m)]$ *such that*

$$(-1)^{i-1} D_i/D_0 < 0; \qquad i = 1, ..., n. \tag{2.13}$$

Remark 2.1. By the definition of linear sgn-dependence it is clear that Theorem 2.1 and hence Corollary 2.5 remain true if we replace functionals $f_0, ..., f_n$ by $f_{\sigma(0)}, ..., f_{\sigma(n)}$, where $\{\sigma(0), ..., \sigma(n)\} = \{0, ..., n\}$.

It should be noted that Corollary 2.5 is also an immediate consequence of [1, Theorems 4.1 and 6.1]. On the other hand, the "strong uniqueness" Theorem 6.1 is an immediate consequence of [1, Theorem 4.1] and Corollary 2.5. In the particular case $X = C(T)$, the classes of all interpolating and Haar subspaces M coincide [1, Theorem 3.2]. We recall that an n-dimensional subspace $M = \operatorname{span}\{x_1, ..., x_n\}$ of $C(T)$ is called a *Haar subspace* if $\det[x_i(t_j)] \neq 0$ for all pairwise distinct points $t_1, ..., t_n$ in T. In this case, we have additionally $f_i = \operatorname{sgn}[z(t_i)]\, \delta_{t_i}$ $(z = x - m)$ and

$$D_i = G_i \prod_{i \neq j = 0}^{n} \operatorname{sgn}[z(t_j)],$$

where $G_i = \det[x_\mu(t_\nu)]_{\nu=0,\ \mu=1}^{n,\ \ n}$ with $\nu \neq i$. Hence the inequalities (2.13) can be rewritten in the form

$$(-1)^{i-1} (G_i/G_0) \operatorname{sgn}[z(t_i)\, z(t_0)] < 0; \qquad i = 1, ..., n. \tag{2.14}$$

Moreover, if M is an n-dimensional Haar subspace of $C[a, b]$, then the functionals $f_i = \operatorname{sgn}[z(t_i)]\, \delta_{t_i}$ in Corollary 2.5 can be rearranged so that $t_0 < \cdots < t_n$, which implies that $G_i/G_0 > 0$ for all i [4]. Hence Corollary 2.5 combined with (2.14) gives the classical alternation characterization of (strongly unique) best approximations in this case.

Finally, we note that the inequality $k \leqslant n$ occurring in Theorem 2.1 cannot be improved in general.

EXAMPLE 2.1. Define the n-dimensional subspace M of the space $C[0, n]$ by $M = \operatorname{span}\{x_1, ..., x_n\}$, where $x_i(t) = s(t - i + 1)$; $0 \leqslant t \leqslant n$, and the function $s\colon \mathbb{R} \to [-1, 1]$ is equal to

$$s(t) = \begin{cases} -4t, & \text{if } 0 \leqslant t \leqslant \frac{1}{4}, \\ 4(t - \frac{1}{2}), & \text{if } \frac{1}{4} < t \leqslant \frac{3}{4}, \\ -4(t-1), & \text{if } \frac{3}{4} < t \leqslant 1, \\ 0, & \text{otherwise.} \end{cases}$$

Then we have $\|x - y\| - \|x\| = \|y\|$, whenever $x(t) \equiv 1$ on $[0, n]$ and $y \in M$. Hence $m = 0$ is a strongly unique best approximation in M to this function x. It is clear that functionals f_i, $g_i \in \operatorname{Ext}[F(x)]$ $(i = 1, ..., n)$ defined by

$$g_i(y) = y(i - \tfrac{3}{4}) \qquad \text{and} \qquad f_i(y) = y(i - \tfrac{1}{4}), \qquad y \in C[0, n],$$

are admissible in Theorem 2.1. On the other hand, it is not difficult to show that this is no longer true for any functionals $f_1, ..., f_n, g_1, ..., g_k \in \operatorname{Ext}[F(x)]$ in the case $k < n$.

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